Midterm

You will have 90 minutes to complete this midterm. Please attempt *all* of the questions. Please do not confer with other students and please turn off all electronic devices before the examination begins. Once you have completed the questions, you may hand your answer booklet in at the front and leave.

Throughout the examination, you may use any results from the lectures or the homework, as long as they are stated clearly.

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(Q1) In this question, $\gamma : \mathbb{R} \to \mathbb{S}^2$ is a smooth regular curve inside the unit sphere

$$\mathbb{S}^2 := \{v \in \mathbb{R}^3 : \|v\| = 1\}.$$

a) Define what it means for the curve γ to be regular.

b) We define another smooth curve $\eta : \mathbb{R} \to \mathbb{R}^3$ via the formula

$$\eta(t) := \int_0^t \gamma(s) \times \gamma'(s)\, ds, \quad \forall t \in \mathbb{R}.$$

Show that, if the curve γ is parameterised by arc-length, then the curve η is also parameterised by arc-length.

c) If κ_γ and κ_η denote the curvatures of γ and η respectively, show that

$$\kappa_\eta(t) \leq \kappa_\gamma(t), \quad \forall t \in \mathbb{R}.$$

Mistake in these questions full marks

d) Show that the torsion of η vanishes.

e) In the situation that the torsion of γ is zero, show that η is a straight line in $\mathbb{R}^3$.

(Q2) In this question, we fix a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$, and for each $\theta \in \mathbb{R}$, define the smooth function $g_\theta : \mathbb{R}^3 \to \mathbb{R}$ by

$$g_\theta(x, y, z) := f\left(\cos(z\theta)x + \sin(z\theta)y, -\sin(z\theta)x + \cos(z\theta)y\right), \quad \forall (x, y, z) \in \mathbb{R}^3.$$

We then consider the level set of g_θ at height zero

$$S_\theta := \{(x, y, z) \in \mathbb{R}^3 : g_\theta(x, y, z) = 0\}.$$

a) Define what it means for $\lambda \in \mathbb{R}$ to be a regular value of f.

b) Find expressions for $\frac{\partial g_\theta}{\partial x}$ and $\frac{\partial g_\theta}{\partial y}$ in terms of the partial derivatives of f, and the trignometric functions $\sin(z\theta)$ and $\cos(z\theta)$.

c) If 0 is a regular value of f, show that S_θ is a regular surface.

d) Define what it means for a pair of regular surfaces to be diffeomorphic.

e) Show that the regular surfaces $\{S_\theta : \theta \in \mathbb{R}\}$ are pairwise diffeomorphic.

For $\theta_1, \theta_2 \in \mathbb{R}$, show that $S_{\theta_1}, S_{\theta_2}$ are diffeomorphic.

(Q3) This question is concerning the Catenoid

$$C := \{(x, y, z) \in \mathbb{R}^3 : \cosh^2(z) - x^2 - y^2 = 0\}.$$

a) Show that the Catenoid C is a regular surface.

b) Consider the local coordinates $X : (-\pi, \pi) \times \mathbb{R} \to C$ given by

$$X(u, v) = (\cos u \cosh v, \sin u \cosh v, v), \quad \forall (u, v) \in (-\pi, \pi) \times \mathbb{R}.$$

Find the first fundamental form g with respect to this chart X.

c) Calculate the area of the compact region

$$\Omega := C \cap \{x \geq 0\} \cap \{-1 \leq z \leq 1\}.$$

Hint: $\frac{d}{dx}(x + \sinh x \cosh x) = 2\cosh^2 x$.

d) Recall, with respect to the local coordinates X, there is a locally well-defined unit normal vector

$$N_{(u,v)} = \frac{X_u \times X_v}{\|X_u \times X_v\|}|_{(u,v)}, \quad \forall (u, v) \in (-\pi, \pi) \times \mathbb{R}.$$

Show that

$$N_{(u,v)} = \left(\frac{\cos u}{\cosh v}, \frac{\sin u}{\cosh v}, -\tanh v \right).$$

e) At a fixed point $(u_0, v_0) \in (-\pi, \pi) \times \mathbb{R}$, calculate the matrix of the shape operator $-dN_{(u_0,v_0)}$ with respect to the basis of $T_{(u_0,v_0)}C$ associated to X.

What can you deduce about the mean curvature of C at this point (u_0, v_0)?

Hint: Calculate the vectors N_u and N_v, and express them with respect to the basis $\{X_u, X_v\}$.

(Q1) In this question, $\gamma : \mathbb{R} \to \mathbb{S}^2$ is a smooth regular curve inside the unit sphere

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d) Show that the torsion of η vanishes.

e) In the situation that the torsion of γ is zero, show that η is a straight line in $\mathbb{R}^3$.

b) Since γ is param. by arc-length and $\gamma(s) \in \mathbb{S}$

$\|\gamma(s)\| \equiv 1$ for all $s \in \mathbb{R}$

Differentiating, we have

$0 = 2\gamma'(s) \cdot \gamma(s)$ for all $s \in \mathbb{R}$

So $\gamma'(s) \perp \gamma(s)$ for all $s \in \mathbb{R}$.

Then since $\|\gamma'(s)\| = 1$,

$\gamma \times \gamma'$ is a unit vector.

Then note by FTC that $\eta'(t) = \gamma(t) \times \gamma'(t)$

c) $K_\eta(t) = \|\eta''(t)\| = \|\gamma(t) \times \gamma''(t)\| \leq \|\gamma(t)\| \|\gamma''(t)\| = K_\gamma(t)$.

(with $\|\gamma(t)\| = 1$ and $\|\gamma''(t)\| = K_\gamma(t)$)

(Q2) In this question, we fix a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$, and for each $\theta \in \mathbb{R}$, define the smooth function $g_\theta : \mathbb{R}^3 \to \mathbb{R}$ by

$$g_\theta(x, y, z) := f(\cos(z\theta)x + \sin(z\theta)y, -\sin(z\theta)x + \cos(z\theta)y), \quad \forall (x, y, z) \in \mathbb{R}^3.$$

We then consider the level set of g_θ at height zero

$$S_\theta := \{(x, y, z) \in \mathbb{R}^3 : g_\theta(x, y, z) = 0\}.$$

a) Define what it means for $\lambda \in \mathbb{R}$ to be a regular value of f.

b) Find expressions for $\frac{\partial g_\theta}{\partial x}$ and $\frac{\partial g_\theta}{\partial y}$ in terms of the partial derivatives of f, and the trignometric functions $\sin(z\theta)$ and $\cos(z\theta)$.

c) If 0 is a regular value of f, show that S_θ is a regular surface.

d) Define what it means for a pair of regular surfaces to be diffeomorphic.

e) Show that the regular surfaces $\{S_\theta : \theta \in \mathbb{R}\}$ are pairwise diffeomorphic.

b) $f(x,y,z) = (f_1(x,y,z), f_2(x,y,z))$

where $f_1(x,y,z) = \cos(z\theta)x + \sin(z\theta)y$

$f_2(x,y,z) = -\sin(z\theta)x + \cos(z\theta)y$.

$$\frac{\partial g_\theta}{\partial x} = f_1 \cos(z\theta) - f_2 \sin(z\theta)$$

$$\frac{\partial g_\theta}{\partial y} = f_1 \sin(z\theta) + f_2 \cos(z\theta).$$

c) WTS 0 is a regular value of g_θ

B/c. $S_\theta = g_\theta^{-1}(0)$.

Sps that 0 is not a regular value of g_θ. Then there is a $p \in (x_0, y_0, z_0)$ with $g_\theta(p) = 0$, $dg_{\theta_p} = 0$.

Let $q = (\cos(z_0\theta)x_0 + \sin(z_0\theta)y_0, -\sin(z_0\theta)x_0 + \cos(z_0\theta)y_0) \in \mathbb{R}^2$ $\quad f(q) = g_\theta(p) = 0$.

Since $0 = \left(\frac{\partial g_\theta}{\partial x}(p)\right)^2 + \left(\frac{\partial g_\theta}{\partial y}(p)\right)^2 = f_1^2(q) + f_2^2(q)$

This gives $df_q = 0$ But this is a contradiction to the fact that 0 is a regular value of f.

e) For $\theta, \beta \in \mathbb{R}$, construct diffeomorphism $\varphi: S_\beta \to S_\theta$

WLOG I can take $\beta = 0$. i.e. WT find diffeomorphism $\varphi: S_0 \to S_\theta$

$g_0(x,y,z) = f(x,y)$ So notice that $g_0 = f \circ \pi$ where π is projection from $\mathbb{R}^3$ to xy-plane.

Take $\varphi(x,y,z) = (\cos(z\theta)x + \sin(z\theta)y, -\sin(z\theta)x + \cos(z\theta)y, z)$

φ is clearly smooth as a map from $\mathbb{R}^3$ to $\mathbb{R}^3$.

with smooth inverse $\varphi^{-1}(x,y,z) = (\cos(-z\theta)x + \sin(-z\theta)y, -\sin(-z\theta)x + \cos(-z\theta)y, z)$.

as a map from $\mathbb{R}^3 \to \mathbb{R}^3$

So suffices to show $\varphi(S_0) = S_\theta$.

Let $(x,y,z) \in S_\theta$. So $g_\theta(x,y,z) = 0$

$\Leftrightarrow f(\cos(z\theta)x + \sin(z\theta)y, -\sin(z\theta)x + \cos(z\theta)) = 0$

$\Leftrightarrow f(\pi(\varphi(x,y,z))) = 0$

$\Leftrightarrow (f \circ \pi \circ \varphi)(x,y,z) = 0$

$\Leftrightarrow (g_0 \circ \varphi)(x,y,z) = 0$. So $g_0(\varphi(x,y,z)) = 0$.

So $\varphi(x,y,z) \in S_0$

Similarly for φ^{-1}.

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a) Show that the Catenoid C is a regular surface.

b) Consider the local coordinates $X : (-\pi, \pi) \times \mathbb{R} \to C$ given by

$$X(u, v) = (\cos u \cosh v, \sin u \cosh v, v), \quad \forall (u, v) \in (-\pi, \pi) \times \mathbb{R}.$$

Find the first fundamental form g with respect to this chart X.

c) Calculate the area of the compact region

$$\Omega := C \cap \{x \geq 0\} \cap \{-1 \leq z \leq 1\}.$$

Hint: $\frac{d}{dx}(x + \sinh x \cosh x) = 2\cosh^2 x.$

a) $f(x,y,z) = \cosh^2(z) - x^2 - y^2$.

Then $C = f^{-1}(0)$.

Need to check 0 is a regular value of f.

So $f(0,0,0) = 1$.

$df_{(x,y,z)} = (-2x, -2y, 2\cosh z \sinh(z))$

So we see that $df_{(x,y,z)} = 0$ only at $(x,y,z) = (0,0,0)$.

So 0 is a regular value and C is a regular surface.

b) $X_u = (-\sin u \cosh v, \cos u \cosh v, 0)$

$X_v = (\cos u \sinh v, \sin u \sinh v, 1)$

So $g_{11} = X_u \cdot X_u = \cosh^2 v \sin^2 u + \cosh^2 v \cos^2 u = \cosh^2 v.$

$g_{12} = X_u \cdot X_v = 0$

$g_{21} = X_v \cdot X_u = 0$

$$g_{22} = X_v \cdot X_v = \cos^2 u \sinh^2 v + \sin^2 u \sinh^2 v + 1$$
$$= 1 + \sinh^2 v = \cosh^2 v.$$

So $g = \begin{bmatrix} \cosh^2 v & 0 \\ 0 & \cosh^2 v \end{bmatrix}.$

c) $$A = \int_\Omega dA = \int_\Omega \sqrt{\det g}\, du\, dv = \int_{-1}^{1} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cosh^2 v \, du\, dv = \pi \int_{-1}^{1} \cosh^2 v \, dv$$

$$= \frac{\pi}{2} \left(v + \sinh v \cosh v \right) \Big|_{-1}^{1} = \pi + \pi \sinh(1) \cosh(1).$$

d) Recall, with respect to the local coordinates X, there is a locally well-defined unit normal vector

$$N_{(u,v)} = \frac{X_u \times X_v}{\|X_u \times X_v\|}|_{(u,v)}, \quad \forall (u,v) \in (-\pi,\pi) \times \mathbb{R}.$$

Show that

$$N_{(u,v)} = \left(\frac{\cos u}{\cosh v}, \frac{\sin u}{\cosh v}, -\tanh v\right).$$

e) At a fixed point $(u_0, v_0) \in (-\pi,\pi) \times \mathbb{R}$, calculate the matrix of the shape operator $-dN_{(u_0,v_0)}$ with respect to the basis of $T_{(u_0,v_0)}C$ associated to X.

What can you deduce about the mean curvature of C at this point (u_0, v_0)?

Hint: Calculate the vectors N_u and N_v, and express them with respect to the basis $\{X_u, X_v\}$.

d) $X_u \times X_v = (\cos u \cosh v, \sin u \cosh v, -\sinh v \cosh v)$

$\|X_u \times X_v\| = \cosh^2 v.$

$N = \frac{X_u \times X_v}{\|X_u \times X_v\|} = \left(\frac{\cos u}{\cosh v}, \frac{\sin u}{\cosh v}, -\sinh(v)\right)$

e) At pt. (u_0, v_0), we have

$N_u = \frac{1}{\cosh v_0}(-\sin u_0, \cos u_0, 0) = \frac{1}{\cosh^2 v_0} X_{u_0}$

$N_v = \frac{-1}{\cosh^2 v_0}(\cos u_0 \sinh v_0, \sin u_0 \sinh v_0, 1) = \frac{-1}{\cosh^2 v_0} X_{v_0}.$

So in basis $\{X_{u_0}, X_{v_0}\}$, we have

$$-dN_{(u_0,v_0)} = \begin{bmatrix} \frac{-1}{\cosh^2 v_0} & 0 \\ 0 & \frac{1}{\cosh^2 v_0} \end{bmatrix}$$

$$H = \frac{1}{2}\left(\frac{-1}{\cosh^2 v_0} + \frac{1}{\cosh^2 v_0}\right) = 0.$$

i.e. catenoid is a minimal surface. $(H \equiv 0)$.